AP CALCULUS AB
TOPIC 3: POLYNOMIAL THEOREMS

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1. Preliminaries

1.1. Basic Definitions.

Definition 1. A *polynomial* with real coefficients is a function of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,$$

where $a_i \in \mathbb{R}$ for $i = 0, \dots, n$, and $a_n \neq 0$ unless $f(x) = 0$.

We call n the *degree* of f.

We call the a_i's the *coefficients* of f.

We call a_0 the *constant coefficient* of f, and set $\mathrm{CC}(f) = a_0$.

We call a_n the *leading coefficient* of f, and set $\mathrm{LC}(f) = a_n$.

We say that f is *monic* if $\mathrm{LC}(f) = 1$.

The *zero function* is the polynomial of the form $f(x) = 0$.

A *constant function* is a polynomial of degree zero, so it is of the form $f(x) = c$ for some $c \in \mathbb{R}$. The graph of a constant function is a horizontal line. Constant polynomials may be viewed simply as real numbers.

A *linear function* is a polynomial of degree one, so it is of the form $f(x) = mx+b$ for some $m, b \in \mathbb{R}$ with $m \neq 0$. The graph of such a function is a non-horizontal line.

A *quadratic function* is a polynomial of degree two, of the form $f(x) = ax^2+bx+c$ for some $a, b, c \in \mathbb{R}$ with $a \neq 0$.

A *cubic function* is a polynomial of degree three.

A *quartic function* is a polynomial of degree four.

A *quintic function* is a polynomial of degree five.

1.2. Basic Facts.

Let f and g be real valued functions of a real variable. We can define the addition, subtraction, product, and quotient of these functions pointwise; for example, $(f+g)(x) = f(x) + g(x)$ for all $x \in \mathrm{dom}(f) \cap \mathrm{dom}(g)$.

The sum, difference, product, and composition of two polynomials is also a polynomial, and we can determine its degree as follows.

- $\deg(f+g) = \max\{\deg(f), \deg(g)\}$
- $\deg(f \cdot g) = \deg(f) + \deg(g)$
- $\deg(f \circ g) = \deg(f) \cdot \deg(g)$

An exception to the first rule exists when $\deg(f) = \deg(g)$ and $\mathrm{LC}(f) - \mathrm{LC}(g) = 0$. The quotient of two polynomials is normally not a polynomial; it is a rational function.

Date: August 17, 2022.

2. Quadratic Polynomials

2.1. **The Quadratic Formula.** A *quadratic polynomial* is a polynomial of degree two. It is traditional to write this as

$$f(x) = ax^2 + bx + c.$$

The *zeros* of a function f are the solutions to the equation $f(x) = 0$. If $a \in \mathbb{R}$ and $f(a) = 0$, then $(a, 0)$ is an x-intercept of f; thus finding zeros is important for graphing. One of the most useful theorems in mathematics provides a formula to find the zeros of a quadratic function.

Theorem 1 (Quadratic Formula). *Let $f(x) = ax^2 + bx + c$. Then the solutions to the equation $f(x) = 0$ are given by*

$$\boxed{x = \frac{b - \sqrt{b^2 - 4ac}}{2a}.}$$

Proof. The equation $f(x) = 0$ can be solved using the method of completing the square:

$$\begin{aligned} ax^2 + bx + c = 0 &\Rightarrow x^2 + \frac{b}{a} = -\frac{c}{a} \\ &\Rightarrow x^2 + \frac{b}{a} + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{c}{a} \\ &\Rightarrow \left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2} \\ &\Rightarrow x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a} \\ &\Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \end{aligned}$$

□

2.2. **The Discriminant.** The *discriminant* of $f(x) = ax^2 + bx + c$ is

$$\Delta = b^2 - 4ac.$$

This is the quantity under the radical, and so determines the number of real solutions the equation $f(x) = 0$.

- If $\Delta > 0$, then $f(x) = 0$ has two distinct real solutions.
- If $\Delta = 0$, then $f(x) = 0$ has a unique real solution.
- If $\Delta < 0$, then $f(x) = 0$ has no real solutions.

If $\Delta < 0$, the solutions are complex; we will investigate this later.

3. Polynomial Division

3.1. **The Division Algorithm.** The *Division Algorithm for Integers* states that for every $m, n \in \mathbb{Z}$, there exist unique integers $q, r \in \mathbb{Z}$ such that

$$n = mq + r \quad \text{and} \quad 0 \leq r < m.$$

We call n the *dividend*, m is the *divisor*, q is the *quotient*, and r is the *remainder*. We know this is true, because we know how to find q and r; we use long division to divide m into n.

An analogous situation exists for polynomials, which we state in the form of a theorem.

Theorem 2 (Division Algorithm for Polynomials)**.** *Let f and g be polynomials with real coefficients. Then there exist polynomials q and r with real coefficients such that*

$$g = fq + r \quad \textit{and} \quad \deg(r) < \deg(f).$$

Reason. Use long division to divide f into g. This process stops when the remainder is of degree less than that of f. Let q be the quotient and let r be the remainder. □

3.2. **The Remainder Theorem.** We divide by a linear polynomial and use the division algorithm to show that remainders are values of the dividend.

Theorem 3 (Remainder Theorem)**.** *Let $a \in \mathbb{R}$ and set $f(x) = x - a$. Then f is a polynomial with real coefficients. Let g be another polynomial with real coefficients, are write $g = fq + r$, with $\deg(r) < \deg(f)$. Then $r \in \mathbb{R}$, and $g(a) = r$.*

Proof. A polynomial of degree 0 is a real number, and since $\deg(r) < \deg(f) = 1$, we see that $\deg(r) = 0$, so $r \in \mathbb{R}$. Thus $g(x) = f(x)q(x) + r$ for every $x \in \mathbb{R}$. Plug in $x = a$ to see that

$$\begin{aligned} g(a) &= f(a)q(a) + r \\ &= (a - a)q(a) + r && \text{because } f(x) = (x - a) \\ &= 0 \cdot q(a) + r \\ &= r \end{aligned}$$

□

3.3. **The Factor Theorem.** If $m, n \in \mathbb{Z}$, we say that m *divides* n, and write $m \mid n$, if there exists $k \in \mathbb{Z}$ such that $n = km$. That is, m divides n if m is a divisor of n, or if m is a factor of n, or if n is a multiple of m. These definitions have direct analogues for polynomials.

Definition 2. Let f and g be polynomials with real coefficients. We say that f *divides* g, and write $f \mid g$, if there exists a polynomial k such that $g = kf$. In this case, we may say that f is a factor of g, or that g is a multiple of f.

We provide the precise condition under which a linear polynomial is a factor of another polynomial.

Theorem 4 (Factor Theorem)**.** *Let g be a polynomial with real coefficients and let $a \in \mathbb{R}$. Set $f(x) = x - a$. Then $g(a) = 0$ if and only if $f \mid g$.*

Proof. Suppose that $f \mid g$. Then, $g = kf$ for some polynomial k. Then $g(x) = k(x)f(x)$ for every $x \in \mathbb{R}$. Thus $g(a) = k(a)f(a) = k(a)(a - a) = k(a) \cdot 0 = 0$.

On the other hand, suppose that $g(a) = 0$. Then 0 is the remainder when g is divided by f; that is, if q is the quotient and r is the remainder when g is divided by f, we have

$$g = fq + r = fq + 0 = fq,$$

so $f \mid g$. □

The Factor Theorem says that the zeros of g produce linear factors of g, and vice versa. Thus in order to find the x-intercepts of a polynomial g, we factor it.

Example 1. Find all x-intercepts of the graph of $g(x) = x^3 - 3x^2 + x - 3$.

Solution. We use the technique of "factoring by grouping" to see that

$$g(x) = (x - 3)(x^2 + 1).$$

Since $x^2 + 1 = 0$ has no real solutions, the Factor Theorem tells us that the only real zero of f is $x = 3$. Thus the only x-intercept of f is $(3, 0)$. □

3.4. **Synthetic Division.** Let g be a polynomial with real coefficients and let $a \in \mathbb{R}$. Set $f(x) = x - a$. The process of dividing f into g can be written with less notation by using *synthetic division*, which will be demonstrated in class. In essence, this consists of writing the division tableau without writing down any of the x's.

It is essential to realize that synthetic division produces both the quotient and the remainder of g divided by f, and that the remainder theorem dictates that the remainder is the value of f at a.

It is worth noting that synthetic division is the exact process used by efficient computer algorithms to evaluate polynomials, although this is not expressly stated. We can rewrite the polynomial in *Horner's form*:

$$a_0 + a_1 x + \cdots + a_{n-1}x^{n-1} + a_n x^n = a_0 + x(a_1 + x(a_2 + \cdots + x(a_{n-1} + x a_n) \cdots).$$

Now evaluating Horner's form is identical to performing synthetic division and applying the remainder theorem to obtain the value of the polynomial.

Example 2. Let $f(x) = x^4 + 2x^3 + 3x^2 + 4x + 5$. Then $f(x) = 5 + x(4 + x(3 + x(2 + x)))$. For a given x, evaluating the first form four additions and nine multiplications, whereas evaluating the second form requires four additions but only three multiplications.

3.5. **Irreducible Polynomials.**

Definition 3. Let g be a nonconstant polynomial with real coefficients. We say that g is *reducible* over $\mathbb{R}$ if there exist polynomials f, k with real coefficients and of lower degree than g, such that $g = kf$. Otherwise, g is *irreducible* over $\mathbb{R}$.

We say that g is *factored completely* over $\mathbb{R}$ if it its written as a product of polynomials which are irreducible over $\mathbb{R}$.

If we can factor a polynomial completely, the Factor Theorem says that this will produce *all* of the zeros of the polynomial.

It is obvious that any polynomial can be factored into the product of a constant polynomial and a polynomial of the same degree. A factorization of a polynomial into polynomials of lower degree is sometimes called a *nontrivial factorization*.

If f is a quadratic polynomial with real coefficients, any nontrivial factorization produces linear factors, which by the Factor Theorem produce real zeros. But if the discriminant is negative, then f has no real zeros, so we see that f is irreducible if and only if its discriminant is zero.

We have a similar idea with respect to factorization over the rationals.

Definition 4. Let g be a polynomial with rational coefficients. We say that g is *reducible* over $\mathbb{Q}$ if there exist polynomials f, k with rational coefficients and of lower degree than g, such that $g = kf$. Otherwise, g is *irreducible* over $\mathbb{Q}$.

In this case, if f is a quadratic polynomial with integer (or rational) coefficients, it is reducible over $\mathbb{Q}$ if and only if the discriminant Δ is a perfect square. Otherwise, although f has a nontrivial factorization over $\mathbb{R}$, it has none over $\mathbb{Q}$.

4. Rational Zeros

Definition 5. Let f be a polynomial of degree n with real coefficients. Then there exist real numbers $a_0, a_1, \ldots, a_n$, with $a_n \neq 0$, such that

$$f(x) = a_n x^n + \cdots + a_1 x + a_0.$$

The *constant coefficient* of f is a_0, and is denoted $\mathrm{CC}(f)$.

The *leading coefficient* of f is a_n, and is denoted $\mathrm{LC}(f)$.

That is,

$$\mathrm{CC}(f) = a_0 \quad \text{and} \quad \mathrm{LC}(f) = a_n.$$

Notice that $(x-2)(x-3) = x^2 - 5x + 6$. It is easy to see that the constant coefficient 6 must be the product of the constant coefficients 2 and 3 of the linear factors. We generalize this with a theorem on rational zeros. First we note that if $a \in \mathbb{Q}$, then there exist integers p, q such that $a = \dfrac{p}{q}$. We can pick p and q uniquely by insisting that q is positive, and that it has no common factors with p. Let $\gcd(p, q)$ denote the *greatest common divisor* of p and q; this is the largest common divisor of p and q.

Theorem 5 (Rational Zeros Theorem)**.** *Let f be a polynomial with integer coefficients, and let $a \in \mathbb{Q}$. Then there exist integers $p, q \in \mathbb{Z}$ with $a = \frac{p}{q}$ such that $\gcd(p, q) = 1$. If $f(a) = 0$, then $p \mid \mathrm{CC}(f)$ and $q \mid \mathrm{LC}(f)$.*

Proof. Write f as
$$f(x) = a_n x^n + \cdots + a_1 x + a_0.$$
Plug in $x = \frac{p}{q}$ to get
$$a_n \frac{p}{q}^n + \cdots + a_1 \frac{p}{q} + a_0 = 0.$$
Clear denominators by multiplying though by q^n to get
$$a_n p^n + a_{n-1} p^{n-1} q + \cdots + a_1 p q^{n-1} + a_0 q^n = 0.$$

Now
$$a_0 q^n = -p(a_n p^{n-1} + a_{n-1} p^{n-2} q + \cdots + a_1 q^{n-1}).$$
Since p divides the right hand side, is must also divide the left hand side; however, since p and q have no common factors, we see that p divides a_0.

Similarly,
$$a_n p^n = -q(a_{n-1} p^{n-1} + \cdots + a_1 p q^{n-2} + a_0 q^{n-1}).$$
Since q divides the right hand side, is must also divide the left hand side; however, since p and q have no common factors, we see that q divides a_n. □

Example 3. Factor $f(x) = 2x^3 - x^2 - x - 3$.

Proof. By the Rational Zeros Theorem, the only possibly rational zeros are
$$\pm 1,\ \pm 3,\ \pm\frac{1}{2},\ \text{and}\ \pm\frac{3}{2}.$$
We use synthetic division to test these one at a time; we find that $f(\frac{3}{2}) = 0$. Thus $f(x) = (x - \frac{3}{2})(2x^2 + 2x + 2) = (2x - 3)(x^2 + x + 1)$. Since $\Delta(x^2 + x + 1) = -3 < 0$, this is a complete factorization over $\mathbb{R}$. □

5. Complex Numbers

5.1. Basic Definitions.

Definition 6. A *complex number* is an expression of the form $x+iy$, where $x, y \in \mathbb{R}$ and i is a new symbol such that $i^2 = -1$. Let $\mathbb{C}$ denote the set of all complex numbers, so that

$$\mathbb{C} = \{x + iy \mid x, y \in \mathbb{R} \text{ and } i^2 = -1\}.$$

We can add and multiply complex numbers by treating i like a variable, combining like terms, and replacing all occurrences of i^2 with -1. Thus, if $z_1 = x_1 + iy^1$ and $z_2 = x_2 + iy_2$, then

- $z_1 + z_2 = (x_1 + x_2) + i(y + 1 + y_2)$;
- $z_1 \cdot z_2 = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)$.

The *real part* of $z = x + iy$ is

$$\Re(z) = x.$$

The *imaginary part* of $z = x + iy$ is

$$\Im(z) = y.$$

The *modulus* (aka length, magnitude, or norm) of $z = x + iy$ is

$$|z| = \sqrt{x^2 + y^2}.$$

Note that $|z| \in \mathbb{R}$.

The *conjugate* of $z = x + iy$ is

$$\overline{z} = x - iy.$$

We have the following properties of conjugate.

- $|\overline{z}| = |z| \in \mathbb{R}$
- $z + \overline{z} = 2\Re(z) \in \mathbb{R}$
- $z\overline{z} = x^2 + y^2 = |z|^2 \in \mathbb{R}$

We can use the conjugate to divide complex numbers. Thus if $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, then

$$\frac{z_1}{z_2} = \frac{z_1\overline{z_2}}{z_2\overline{z_2}} = \frac{(x_1x_2 + y_1y_2) + i(x_2y_1 - x_1y_2)}{x_2^2 + y_2^2} = \frac{(x_1x_2 + y_1y_2)}{x_2^2 + y_2^2} + i\frac{x_2y_1 - x_1y_2}{x_2^2 + y_2^2}.$$

The result has been put in the *standard form* of a complex number, which is $a + ib$ where $a, b \in \mathbb{R}$.

It is important to understand that real number are complex numbers; if $x \in \mathbb{R}$, then $x = x + i \cdot 0$, so that $x = \Re(x)$.

We have

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}.$$

5.2. **Relationship to Quadratic Functions.** It is traditional to denote the independent variable over $\mathbb{C}$ by the letter z. We now adopt that convention.

Let $f(z) = az^2+bz+c$, where $a, b, c \in \mathbb{R}$. If $\Delta(f) < 0$, then f has no real zeros. In this case, however, we consider that $\sqrt{b^2 - 4ac} = i\sqrt{4ac - b^2}$, where $\sqrt{4ac - b^2} \in \mathbb{R}$. Then the zeros of a a quadratic function with negative discriminant may be view and complex zeros.

Thus let $w = \frac{-b}{2} + i\frac{\sqrt{4ac - b^2}}{2}$, so that $f(w) = 0$. Then $\overline{w} = \frac{-b}{2} - i\frac{\sqrt{4ac - b^2}}{2}$ is the other zero of f, and f factors as

$$f(z) = a(z - w)(z + w).$$

On the other hand, let $w \in \mathbb{C}$ be an arbitrary complex number, and set

$$\begin{aligned} f(z) &= (z - w)(z + w) \\ &= z^2 - (w + \overline{w})z + w\overline{w} \\ &= z^2 - 2\Re(w)z + |w|^2 \end{aligned}$$

Then f is a polynomial with real coefficients. Note that f is cannot be factored over $\mathbb{R}$; that is, f is an *irreducible quadratic*. Every complex number is the zero of an irreducible quadratic polynomial. Are there any polynomials of degree three or greater which are irreducible over $\mathbb{R}$?

5.3. **The Fundamental Theorem of Algebra.** We can define polynomials with complex coefficients in the same manner as polynomials over $\mathbb{R}$; we simply allow the coefficients to be complex. For example, $x^2 - i$ is a polynomial with complex coefficients. Since real numbers are complex, we view the set of polynomials with real coefficients as a subset of the set of polynomials with complex coefficients. So when we talk about a polynomial with complex coefficients, we are not ruling out the possibility that the polynomial has real, or even integer, coefficients.

The Division Algorithm, Remainder Theorem, and Factor Theorem all remain true for polynomials with complex coefficients.

Complex number were discovered in the fifteenth century, and it quickly became apparent that they were a useful idea for understanding the factors of polynomials with real coefficients. It was long suspected that every polynomial equation has a solution in the complex numbers, but this remained unproven until Gauss supplied a (partial) proof in his doctoral thesis.

Theorem 6 (Fundamental Theorem of Algebra)**.** *Let f be a nonconstant polynomial with complex coefficients. Then there exists $a \in \mathbb{C}$ such that $f(a) = 0$.*

There are several proofs of the Fundamental Theorem, but each of them is too deep for our consideration at this point. So we accept the Fundamental Theorem as given, and move forward to understand how completely polynomials with real or complex coefficients factor. The latter case is relatively easy, as follows.

Theorem 7 (Complete Factorization Theorem)**.** *Let f be a nonconstant polynomial with complex coefficients. Then f is a product of linear factors.*

Proof. By the fundamental theorem, there exists $a_1 \in \mathbb{C}$ such that $f(a_1) = 0$. By the Factor Theorem, $f(z) = (z - a_1)f_2(z)$ for some polynomial f_2 with complex coefficients.

Similarly, f_2 has a complex zero, say $f_2(a_2) = 0$, so that $f_2 = (x - a_2)f_3(z)$ for some polynomial f_3. Thus $f(z) = (z - a_1)(z - a_2)f_3(z)$.

Continuing in this way, we see that there exist $a_1, a_2, \ldots, a_n \in \mathbb{C}$ such that

$$f(z) = (z - a_1)(z - a_2) \cdots (z - a_n),$$

where $n = \deg(f)$. It should be noted that the zeros a_1 through a_n are not necessarily distinct. $\square$

So, polynomials with complex coefficients factor completely into linear factors. That is, the only polynomials with are irreducible over $\mathbb{C}$ are linear. We now proceed to analyze how completely it is possible to factor polynomials over $\mathbb{R}$. Conjugates pairs play a large role in this, so we begin with a lemma regarding conjugation.

Lemma 1. *Let $z_1, z_2 \in \mathbb{C}$. Then $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$.*

Proof. Write $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ where $x_1, y_1, x_2, y_2 \in \mathbb{R}$. Then

$$\begin{aligned} \overline{z_1 + z_2} &= \overline{(x_1 + x_2) + i(y_1 + y_2)} \\ &= (x_1 + x_2) - i(y_1 + y_2) \\ &= x_1 - iy_1 + x_2 - iy_2 \\ &= \overline{z_1} + \overline{z_2}. \end{aligned}$$

$\square$

Lemma 2. *Let $z_1, z_2 \in \mathbb{C}$. Then $\overline{z_1 \cdot z_2} = \overline{z_1} \cdot \overline{z_2}$.*

Proof. Write $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ where $x_1, y_1, x_2, y_2 \in \mathbb{R}$. Then

$$\begin{aligned} \overline{z_1} \cdot \overline{z_2} &= (x_1 - iy_1)(x_2 - iy_2) \\ &= x_1x_2 - ix_1y_2 - iy_1x_2 + i^2 y_1y_2 \\ &= (x_1x_2 - y_1y_2) - i(x_1y_2 + x_2y_1) \\ &= \overline{(x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)} \\ &= \overline{z_1 \cdot z_2} \end{aligned}$$

$\square$

Lemma 3. *Let $z \in \mathbb{C}$. Then $z \in \mathbb{R}$ if and only if $\overline{z} = z$.*

Proof. Let $z = x + iy$. If $z \in \mathbb{R}$, this means that $y = 0$, so $\overline{z} = x - i \cdot 0 = x$, so $z = \overline{z}$. On the other hand, if $\overline{z} = z$, then $x - iy = x + iy$, so $-iy = iy$, so $y = 0$. $\square$

Lemma 4. *Let f be a polynomial with real coefficients, and let $z \in \mathbb{C}$ Then $\overline{f(z)} = f(\overline{z})$.*

Proof. Recall that if $x \in \mathbb{C}$, then $x \in \mathbb{R}$ if and only if $x = \overline{x}$.

Write $f(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$, where $a_n, \ldots, a_0 \in \mathbb{R}$. Now

$$\begin{aligned}
\overline{f(z)} &= \overline{a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0} \\
&= \overline{a_n z^n} + \overline{a_{n-1} z^{n-1}} + \cdots + \overline{a_1 z} + \overline{a_0} && \text{by Lemma 1} \\
&= \overline{a_n}\,\overline{z}^n + \overline{a_{n-1}}\,\overline{z}^{n-1} + \cdots + \overline{a_1}\,\overline{z} + \overline{a_0} && \text{by Lemma 2} \\
&= a_n \overline{z}^n + a_{n-1} \overline{z}^{n-1} + \cdots + a_1 \overline{z} + a_0 && \text{by Lemma 3} \\
&= f(\overline{z})
\end{aligned}$$

□

Theorem 8 (Conjugate Pairs Theorem)**.** *Let f be a polynomial with real coefficients, and let $a \in \mathbb{C}$ If $f(a) = 0$, then $f(\overline{a}) = 0$.*

Proof. Since $0 \in bR$, $\overline{0} = 0$. Thus $f(\overline{a}) = \overline{f(a)} = \overline{0} = 0$. □

Theorem 9 (Linear and Quadratic Factors Theorem)**.** *Every polynomial with real coefficients can be factored into a product of linear and irreducible quadratic factors.*

Proof. We have previously seen that if $w \in \mathbb{C}$, then $(z - w)(z - \overline{w})$ is a polynomial with real coefficients.

Let f be a polynomial with real coefficients. By the Complete Factorization Theorem,

$$f(z) = (z - a_1)(z - a_2) \cdots (z - a_n),$$

where $n = \deg(f)$. By the preceding theorem, the complex zeros occur in conjugate pairs; we can multiply the factors corresponding to each such pair to et a quadratic factor with real coefficients. This results in f being factored into linear and irreducible quadratic factors. □

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